INSTRUCTIONS TO CANDIDATES

The candidate should have the following for this examination:
Answer booklet;
Mathematical table/Scientific calculator;
Drawing instruments.
Answer FIVE of the EIGHT questions in the answer booklet provided.
All questions carry equal marks.
Maximum marks for each part of a question are as shown.
Candidates should answer the questions in English.

This paper consists of 4 printed pages.

Candidates should check the question paper to ascertain that all
the pages are printed as indicated and that no questions are missing.
1. (a) (i) Given that \( x_n \) is an approximation to the root of the equation \( x^3 + 6x - 5 = 0 \), use the Newton-Raphson method to show that a better approximation is given by:

\[
x_{n+1} = \frac{2x_n^3 + 5}{3x_n^2 + 6}.
\]

(ii) Taking \( x_0 = 0.5 \), determine the root, correct to four decimal places. (9 marks)

(b) A polynomial function \( f(x) \) is represented by the data in Table 1.

<table>
<thead>
<tr>
<th>x</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>-2</td>
<td>-1</td>
<td>2</td>
<td>19</td>
<td>62</td>
<td>143</td>
</tr>
</tbody>
</table>

Use the Newton-Gregory forward difference interpolation formula to determine \( f(x) \), and find:

(i) \( f(-3.2) \);

(ii) \( f(5) \). (11 marks)

2. (a) Determine the eigenvalues and corresponding eigenvectors of the matrix:

\[
A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}
\]

(12 marks)

(b) A dynamic system is modelled by the vector matrix differential equation:

\[
\frac{dx}{dt} = Ax, \text{ where } A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \text{ and } x(t) \text{ is the system state vector.}
\]

Determine the system state transition matrix, \( \Phi(t) \). (8 marks)

3. (a) Given the function \( f(z) = z^2 - 2z + 3 \), where the complex variable \( z = x + jy \),

(i) express \( f(z) \) in the form \( u(x,y) + jv(x,y) \);

(ii) show that \( u \) and \( v \) are harmonic functions. (8 marks)

(b) The circle \(|z| = 2\) in the \( z \)-plane is mapped onto the \( w \)-plane by the transformation

\[
w = \frac{z + 3j}{z + j}.
\]
>Determine the centre and radius of the image circle. (12 marks)
4. (a) Find the half-range Fourier cosine series of the function \( f(t) = \pi^2 + t^2, 0 < t < \pi \).
   
   (8 marks)

(b) A function \( f(t) = \begin{cases} 
\pi, & -\pi < t < 0 \\
\pi - t, & 0 < t < \pi \\
f(t + 2\pi) 
\end{cases} \)

Sketch the graph of \( f(t) \) in the interval \(-\pi < t < 3\pi\), and determine its Fourier series representation.
   
   (12 marks)

5. (a) Evaluate the integral

\[
\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{y}{\sqrt{x^2+y^2}} \, dy \, dx
\]

(5 marks)

(b) Change the order of integration, and hence determine the value of the integral:

\[
\int_0^1 \int_0^{1-x} \cos(y^3) \, dy \, dx
\]

(8 marks)

(c) Use a triple integral to find the volume of the solid in the first octant bounded by the coordinate planes, and the plane \( x + y + 2z = 1 \).

(7 marks)

6. Verify Green's theorem in the plane for the line integral \( \oint_C xy \, dx + (x + y^2) \, dy \), where \( C \) is the perimeter of the triangle with vertices \((0,0), (1,1)\) and \((-1,1)\) oriented counterclockwise.

(10 marks)

7. (a) The eigenvalues of a 2 \times 2 matrix \( A \) are \( \lambda_1 = 3 \) and \( \lambda_2 = 2 \) with corresponding eigenvectors \( e_1 = [-2 \ 1]^T \) and \( e_2 = [-1 \ 1]^T \).

Determine:

(i) the modal matrix \( M \) and spectral matrix \( \Lambda \) of \( A \);

(ii) the matrix \( A \);

(iii) \( A^2 \).

(10 marks)

(b) (i) Determine the half-range Fourier sine series of \( f(t) = t, 0 < t < \pi \).

(ii) Use the result in (i) to show that \( \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \).

(10 marks)
8. (a) Evaluate \( \iint_S \mathbf{F} \cdot d\mathbf{S} \), where the vector field \( \mathbf{F} = (y-x)\mathbf{i} + (y-z)\mathbf{j} + (x-z)\mathbf{k} \) and \( S \) is the part of the plane \( x+y+z = 1 \) in the first octant. \( \text{(11 marks)} \)

(b) Use the divergence theorem to evaluate
\[
\iiint_E \nabla \cdot \mathbf{F} \, dV
\]
for the vector field \( \mathbf{F} = -y\mathbf{i} - yz\mathbf{j} + z^2\mathbf{k} \), where \( E \) is the upper hemisphere \( x^2 + y^2 + z^2 = 4, \ z \geq 0 \), oriented by outward unit normals. \( \text{(9 marks)} \)

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